

# A deformation formula for the heat kernel<sup>\*†</sup>

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## Abstract

Let  $P_0$  be a time-dependent partial differential operator acting on functions defined on  $\mathbb{C}^\nu$ , quadratic with respect to  $\partial_{x_1}, \dots, \partial_{x_\nu}, x_1, \dots, x_\nu$ . Let  $c$  be a matrix-valued regular potential. Under suitable conditions, we give an “explicit” expression of “the” heat kernel associated to  $P_0 + c$  for small  $|t|$ ,  $t \in \mathbb{C}$ ,  $\operatorname{Re} t \geq 0$ ,  $x, y \in \mathbb{C}^\nu$ .

## 1 Introduction

Let  $\nu \geq 1$ . For  $j, k = 1, \dots, \nu$ , let  $A_{j,k}$ ,  $B_{j,k}$ ,  $C_{j,k}$  be complex functions analytic in a neighbourhood  $U$  of the origin. Let

$$P_0 := \sum_{j,k=1}^{\nu} A_{j,k}(t) \left( \partial_{x_j} + \sum_{l=1}^{\nu} B_{j,l}(t) x_l \right) \left( \partial_{x_k} + \sum_{l'=1}^{\nu} B_{k,l'}(t) x_{l'} \right) - \sum_{j,k=1}^{\nu} C_{j,k}(t) x_j x_k. \quad (1.1)$$

As well-known, under suitable assumptions on the matrix  $A$ , the equation

$$\begin{cases} \partial_t u = P_0 u \\ u|_{t=0^+} = \delta_{x=y} \end{cases} \quad (1.2)$$

admits an explicit solution

$$p_t^0(x, y) := \frac{1}{(4\pi\Delta t)^{\nu/2}} e^{-\frac{1}{t}\Phi_0(x, y, t)}. \quad (1.3)$$

Here  $\Phi_0$  denotes a polynomial of total degree 2 with respect to  $x, y$ , whose coefficients are analytic near 0;  $\Delta$  is the determinant of the matrix  $(A_{j,k}(0))_{1 \leq j, k \leq \nu}$ .

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Let  $c$  be a regular square matrix-valued function defined on  $U \times \mathbb{C}^\nu$ . Let  $p_t(x, y)$  be a solution of

$$\begin{cases} \partial_t u = (P_0 + c(t, x))u \\ u|_{t=0^+} = \delta_{x=y} \end{cases} \quad (1.4)$$

and let  $p_t^{\text{conj}}(x, y)$  be defined by

$$p_t(x, y) = p_t^0(x, y)p_t^{\text{conj}}(x, y).$$

Roughly speaking, if the autonomous case is considered for the sake of simplicity,  $p_t^{\text{conj}}$  may be related to two transition amplitudes:

$$p_t^{\text{conj}}(x, y) = \frac{\langle y | \exp(t(P_0 + c)) | x \rangle}{\langle y | \exp(tP_0) | x \rangle}.$$

First let us assume that  $P_0 = \partial_{x_1}^2 + \dots + \partial_{x_\nu}^2$  (free case) or, more generally,  $P_0 = \partial_{x_1}^2 + \dots + \partial_{x_\nu}^2 - \lambda(x_1^2 + \dots + x_\nu^2)$ ,  $\lambda \in \mathbb{R}$  (harmonic case). Under strong assumptions on the scalar potential  $c$ ,  $p^{\text{conj}}$  is Borel summable with respect to  $t$  [Ha4]. The same work is done in [Ha5] in the free case, but with a vector potential instead of a scalar one. A so-called deformation formula (respectively its vector potential version) which gives a convenient representation of  $p_t^{\text{conj}}$  is used.

The main aim of this paper is to explore the limits of this formula by considering the non-autonomous case. This explains the choice of the operator  $P_0$  in (1.1). Our deformation formula is given in Theorem 2.1. We do not attempt to give a uniqueness statement for the definition of the heat kernel and we refer to [Ha7] for precise statements about this question.

This formula is related to Wiener and Feynman integrals [It, A-H]. As in these references, we write the potential as the Fourier transform of a Borel measure. See [Ha4] for more details about the relationship between this formula and Wiener or Feynman integrals.

The shape of the formula can be explained by using a heuristic Wiener representation of  $p_t(x, y)$  and Wick's theorem (see [Ha4, Appendix]). However we prove the deformation formula by working directly on the equations satisfied by  $p_t^{\text{conj}}(x, y)$  without attempting to obtain an expression of  $p_t(x, y)$ . This formula uses a so-called deformation matrix. This matrix, in the autonomous case, is considered in [Ge-Ya, On].

By the heuristic method, it is easy to see that the construction of this matrix involves a “propagator” (defined as in quantum field theory). Here we first give another definition of this object and we prove a posteriori that it verifies the propagator equation (see section 3.2).

The shape of the operator  $P_0$  implies that  $p_t^0(x, y)$  can be written explicitly using the solution of a classical Hamiltonian system. The deformation matrix also depends on this solution, hence, by the deformation formula, the expression of  $p_t^{\text{conj}}(x, y)$  only involves objects related to this Hamiltonian system.

We assume in Theorem 2.1 that the functions  $A$ ,  $B$  and  $C$  satisfy a reality assumption (see (2.13)), implying that  $P_0|_{t \in i\mathbb{R}}$  is symmetric with respect to the  $L^2$ -inner product. This assumption is natural if the Schrödinger kernel is considered.

One can certainly establish a deformation formula in the case of a vector potential perturbation of  $P_0$  (instead of a scalar potential one) and give a Borel summability statement for the small time expansion of  $p_t^{\text{conj}}(x, y)$ .

## 2 Notation and main results

For  $z = |z|e^{i\theta} \in \mathbb{C}$ ,  $\theta \in [-\pi/2, \pi/2]$ , let  $z^{1/2} := |z|^{1/2}e^{i\theta/2}$ . For  $T > 0$ , let  $D_T := \{z \in \mathbb{C} \mid |z| < T\}$ ,  $D_T^+ := \{z \in D_T \mid \text{Re}(z) > 0\}$  and  $\bar{D}_T^+ := \{z \in D_T \mid \text{Re}(z) \geq 0\}$ . For  $\lambda, \mu \in \mathbb{C}^\nu$ , we denote  $\lambda \cdot \mu := \lambda_1 \mu_1 + \dots + \lambda_\nu \mu_\nu$ ,  $\lambda^2 := \lambda \cdot \lambda$ ,  $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_\nu)$ ,  $|\lambda| := (\lambda \cdot \bar{\lambda})^{1/2}$  and we extend the two first notations to operators such as  $\partial_x = (\partial_{x_1}, \dots, \partial_{x_\nu})$ . Let  $A = (A_{j,k})_{1 \leq j, k \leq \nu}$  with  $A_{j,k} \in \mathbb{C}$ . If  $x, y \in \mathbb{C}^\nu$ ,  $\sum_{j,k} A_{j,k} x_j y_k$  is denoted by  $x \cdot Ay$  or  $A \cdot x \otimes y$  if  $A$  is symmetric. We set  ${}^t A$  for the transpose of the matrix  $A$  and  $|A|_\infty := \sup_{|x|=1} |Ax|$ . In what follows, we shall consider a potential function defined on  $D_T \times \mathbb{C}^\nu$  with values in a finite dimensional space of square matrices, say  $\mathcal{M}$ . We always use on  $\mathcal{M}$  a norm  $|\cdot|$  such that  $|AB| \leq |A||B|$  for  $A, B \in \mathcal{M}$  and  $|\mathbb{1}| = 1$  ( $\mathbb{1}$  denotes the unitary matrix). Let  $\Omega$  be an open domain in  $\mathbb{C}^m$  and let  $F$  be a complex finite dimensional space. We denote by  $\mathcal{A}(\Omega)$  the space of  $F$ -valued analytic functions on  $\Omega$ , if there is no ambiguity on  $F$ . Let  $T > 0$ .  $\mathcal{C}^\infty(\bar{D}_T^+, \mathcal{A}(\mathbb{C}^{2\nu}))$  denotes the space of smooth functions defined on  $\bar{D}_T^+$  with values in  $\mathcal{A}(\mathbb{C}^{2\nu})$ . We denote by  $\mathcal{C}_{b,1}^\infty(i] - T, T[\times \mathbb{R}^m)$  the space of smooth  $\mathcal{M}$ -valued functions defined on  $i] - T, T[\times \mathbb{R}^m$  such that

$$f \in \mathcal{C}_{b,1}^\infty(i] - T, T[\times \mathbb{R}^m) \Leftrightarrow$$

$$\forall (\alpha, \beta) \in \mathbb{N} \times \mathbb{N}^m, \exists C > 0, \forall (t, x) \in i] - T, T[\times \mathbb{R}^m, |\partial_t^\alpha \partial_x^\beta f(t, x)| \leq C(1 + |x|)^\alpha.$$

We always consider these spaces with their standard Frechet structure (the semi-norms are eventually indexed by compact sets or differentiation order).

Let  $\mathfrak{B}$  be the collection of all Borel sets on  $\mathbb{R}^m$ . An  $F$ -valued measure  $\mu$  on  $\mathbb{R}^m$  is an  $F$ -valued function on  $\mathfrak{B}$  satisfying the classical countable additivity property [Ru]. Let  $|\cdot|$  be a norm on  $F$ . We denote by  $|\mu|$  the positive measure defined by

$$|\mu|(E) = \sup \sum_{j=1}^{\infty} |\mu(E_j)| \quad (E \in \mathfrak{B}),$$

the supremum being taken over all partition  $\{E_j\}$  of  $E$ . In particular  $|\mu|(\mathbb{R}^m) < \infty$ . Note that  $d\mu = h d|\mu|$  where  $h$  is some  $F$ -valued function satisfying  $|h| = 1$   $|\mu|$ -a.e. If  $f$  is an  $F$ -valued measurable function on  $\mathbb{R}^m$  and  $\lambda$  is a positive measure on  $\mathbb{R}^m$  such that  $\int_{\mathbb{R}^m} |f| d\lambda < \infty$ , one may define an  $F$ -valued measure  $\mu$ , by setting  $d\mu = f d\lambda$ . Then  $d|\mu| = |f| d\lambda$ .

Let  $A$ ,  $B$  and  $C$  be some  $\nu \times \nu$  complex matrix-valued analytic functions defined on a neighbourhood of 0 in  $\mathbb{C}$ . Let us assume that the matrices  $A$ ,  $C$  are symmetric and that the matrix  $A(0)$  is real positive definite. The operator  $P_0$  defined in (1.1) can be rewritten as

$$P_0 = A(t) \cdot (\partial_x + B(t)x)^2 - C(t) \cdot x \otimes x \quad (2.1)$$

where

$$A(t) \cdot (\partial_x + B(t)x)^2 := \sum_{j,k=1}^{\nu} A_{j,k}(t) \left( \partial_{x_j} + \sum_{l=1}^{\nu} B_{j,l}(t)x_l \right) \left( \partial_{x_k} + \sum_{l'=1}^{\nu} B_{k,l'}(t)x_{l'} \right).$$

The fact that (1.2) admits a solution as mentioned in the introduction will be recovered later. Our main result gives a formula for the solution of a perturbation of (1.2). We need some classical objects associated to the operator defined in (2.1). For  $t \in \mathbb{C}$ ,  $|t|$  small, let  $L$  be the following Lagrangian acting on  $\mathbb{C}^\nu$ -valued functions

$$L := \frac{1}{4} \dot{q} \cdot A^{-1} \dot{q} + \dot{q} \cdot Bq + q \cdot Cq. \quad (2.2)$$

The Euler-Lagrange equations associated to (2.2) can be written

$$\ddot{q} = E\dot{q} + Fq \quad (2.3)$$

where

$$E := \dot{A}A^{-1} + 2A({}^tB - B), \quad F := 4AC - 2A\dot{B}. \quad (2.4)$$

Let  $x, y \in \mathbb{C}^\nu$ . Let  $t$  be a small positive number. We denote by  $q_t^\natural$  the solution of (2.3) with the conditions  $q_t^\natural(0) = y$ ,  $q_t^\natural(t) = x$  if it is uniquely defined. Notice that  $q_t^\natural$  can be expressed by

$$q_t^\natural = q_t^b x + q_t^\sharp y \quad (2.5)$$

where the matrices-valued functions  $q_t^b$ ,  $q_t^\sharp$  are respectively solutions of

$$\ddot{q} = E\dot{q} + Fq \quad (\text{matrix-valued equation}) \quad (2.6)$$

with the conditions

$$q_t^b(0) = 0, \quad q_t^b(t) = \mathbb{1},$$

respectively

$$q_t^\sharp(0) = \mathbb{1}, \quad q_t^\sharp(t) = 0.$$

Let  $\tilde{q}_t^\diamond$  (with  $\diamond = \natural, b, \sharp$ ) be defined on  $[0, 1]$  by the relation

$$q_t^\diamond(s) = \tilde{q}_t^\diamond\left(\frac{s}{t}\right). \quad (2.7)$$

Notice that  $\tilde{q}_t^b$  is the solution of

$$\ddot{q} = tE(ts)\dot{q} + t^2F(ts)q, \quad q(0) = 0, \quad q(1) = \mathbb{1} \quad (2.8)$$

and that  $\tilde{q}_t^\sharp$  is the solution of

$$\ddot{q} = tE(ts)\dot{q} + t^2F(ts)q, \quad q(0) = \mathbb{1}, \quad q(1) = 0. \quad (2.9)$$

Actually (2.8) (respectively (2.9)) admits a unique solution for complex  $t$  with small modulus, which provides the good definition of  $\tilde{q}_t^\flat(s)$  (respectively  $\tilde{q}_t^\sharp(s)$ ). Then, by (2.7), one gets the existence and uniqueness of  $q_t^\flat(s), q_t^\sharp(s)$  for  $s \in [0, t]$  and small positive  $t$ . The following expressions play a central role in the statement of our main result. For  $s, s' \in [0, t]$ , let  $K_t(s, s')$  be the  $\nu \times \nu$  matrix defined by

$$K_t(s, s') := \int_{s \vee s'}^t q_\tau^\flat(s) A(\tau)^t q_\tau^\flat(s') d\tau. \quad (2.10)$$

For complex  $t$  with small modulus and  $s, s' \in [0, 1]$ , we also denote

$$\tilde{K}_t(s, s') := \int_{s \vee s'}^1 \tilde{q}_{t\tau}^\flat\left(\frac{s}{\tau}\right) A(t\tau)^t \tilde{q}_{t\tau}^\flat\left(\frac{s'}{\tau}\right) d\tau. \quad (2.11)$$

If  $t$  is real, positive and  $s, s' \in [0, t]$ , one gets

$$K_t(s, s') = t\tilde{K}_t\left(\frac{s}{t}, \frac{s'}{t}\right).$$

For  $s = (s_1, \dots, s_n) \in [0, 1]^n$ , we define

$$\tilde{K}_t(s) \cdot \partial_z \otimes_n \partial_z := \sum_{j,k=1}^n \partial_{z_j} \cdot \tilde{K}_t(s_j, s_k) \partial_{z_k}.$$

This differential operator acts on  $\mathcal{A}(\mathbb{C}^{\nu n})$ .

We denote by  $\bar{T}$  a positive number such that (2.8)-(2.9) admits a unique solution for  $t \in D_{\bar{T}}$  and the map  $(t, s) \mapsto q_t^\diamond(s)$  (with  $\diamond = \flat, \sharp$ ) is analytic near  $D_{\bar{T}} \times [0, 1]$ .

**Theorem 2.1** *Let  $T_b > 0$ . Let  $f$  be a measurable function on  $D_{T_b} \times \mathbb{R}^\nu$  with values in a complex finite dimensional space of square matrices, analytic with respect to the first variable. Let  $\mu_*$  be a positive measure on  $\mathbb{R}^\nu$ . Assume that for every  $R > 0$*

$$\int_{\mathbb{R}^\nu} e^{R|\xi|} \sup_{|t| < T_b} |f(t, \xi)| d\mu_*(\xi) < \infty. \quad (2.12)$$

*Let  $\mu_t$  be the measure defined by  $d\mu_t(\xi) = f(t, \xi) d\mu_*(\xi)$  and let*

$$c(t, x) = \int_{\mathbb{R}^\nu} \exp(ix \cdot \xi) d\mu_t(\xi).$$

*Let  $P_0$  be an operator as in (2.1). Let us assume that each  $g = A, iB, C$  satisfies:*

$$\text{The function } g|_{i\mathbb{R}} \text{ is real-valued near } 0. \quad (2.13)$$

Let  $p^{\text{conj}}$  be defined by

$$p^{\text{conj}} = \mathbb{1} + \sum_{n \geq 1} v_n, \quad (2.14)$$

where

$$v_n(t, x, y) := t^n \times \int_{0 < s_1 < \dots < s_n < 1} \left[ e^{t \tilde{\mathbf{K}}_t(s) \cdot \partial_z \otimes_n \partial_z} c(s_n t, z_n) \cdots c(s_1 t, z_1) \right] \Big|_{\substack{z_1 = \tilde{q}_t^{\natural}(s_1) \\ \dots \\ z_n = \tilde{q}_t^{\natural}(s_n)}} d^n s. \quad (2.15)$$

Then there exists  $T_c > 0$  such that

$$p^{\text{conj}} \in \mathcal{A}(D_{T_c}^+ \times \mathbb{C}^{2\nu}) \cap \mathcal{C}^\infty(\bar{D}_{T_c}^+, \mathcal{A}(\mathbb{C}^{2\nu})) \cap \mathcal{C}_{b,1}^\infty[i] - T_c, T_c[\times \mathbb{R}^{2\nu}).$$

The function  $u := p^0 \times p^{\text{conj}}$  is a solution of (1.4).

We shall now give another useful expression of  $p^{\text{conj}}$ . Let  $n \geq 1$ ,  $s = (s_1, \dots, s_n) \in [0, 1]^n$  and  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^{\nu n}$ . Let

$$\begin{aligned} \tilde{q}_t^{\natural}(s) \cdot \xi &:= \tilde{q}_t^{\natural}(s_1) \cdot \xi_1 + \cdots + \tilde{q}_t^{\natural}(s_n) \cdot \xi_n, \\ \tilde{\mathbf{K}}_t(s) \cdot \xi \otimes_n \xi &:= \sum_{j,k=1}^n \xi_j \cdot \tilde{K}_t(s_j, s_k) \xi_k, \\ d^{\nu n} \mu_{st}^{\otimes}(\xi) &:= d^\nu \mu_{s_n t}(\xi_n) \cdots d^\nu \mu_{s_1 t}(\xi_1). \end{aligned} \quad (2.16)$$

Then, we get

**Remark 2.2**

$$v_n(t, x, y) = t^n \int_{0 < s_1 < \dots < s_n < 1} \int e^{i \tilde{q}_t^{\natural}(s) \cdot \xi} e^{-t \tilde{\mathbf{K}}_t(s) \cdot \xi \otimes_n \xi} d^{\nu n} \mu_{st}^{\otimes}(\xi) d^n s. \quad (2.17)$$

**Remark 2.3** Let us consider some examples.

- In the free case  $A = \mathbb{1}$ ,  $B = C = 0$ , we have (see also [Ha4])

$$\tilde{q}_t^{\natural}(s) = y + s(x - y),$$

$$\tilde{\mathbf{K}}_t(s) \cdot \partial_z \otimes_n \partial_z = \sum_{j,k=1}^n s_{j \wedge k} (1 - s_{j \vee k}) \partial_{z_j} \cdot \partial_{z_k}.$$

Then

$$\begin{aligned} v_n(t, x, y) &= t^n \int_{0 < s_1 < \dots < s_n < 1} \int e^{i(y + s(x - y)) \cdot \xi} \times \\ &\quad \exp(-ts(1 - s) \cdot_n \xi \otimes \xi) d^{\nu n} \mu_{st}^{\otimes}(\xi) d^n s \end{aligned}$$

where

$$s(1 - s) \cdot_n \xi \otimes \xi := \sum_{j,k=1}^n s_{j \wedge k} (1 - s_{j \vee k}) \xi_j \cdot \xi_k.$$

- Let us assume that  $A = \mathbb{1}$ ,  $C = 0$  and  $B = -\frac{i}{2}\beta$  where the matrix  $\beta$  is skew-symmetric, real and constant with respect to  $t$ . Then

$$\tilde{q}_t^\natural(s) = e^{-i\beta t(1-s)} \frac{\sin(\beta ts)}{\sin(\beta t)} x + e^{i\beta ts} \frac{\sin(\beta t(1-s))}{\sin(\beta t)} y,$$

$$\tilde{K}_t(s, s') = e^{i\beta t(s'-s)} \frac{\sin(\beta ts \wedge s') \sin(\beta t(1-s \vee s'))}{\beta t \sin(\beta t)}.$$

**Remark 2.4** Let us make some comments on the functional spaces introduced in Theorem 2.1. The space  $\mathcal{A}(D_{T_c}^+ \times \mathbb{C}^{2\nu})$  allows one to consider the function  $p^{\text{conj}}$  as a solution of the (complex) heat equation. The space  $\mathcal{C}^\infty(\bar{D}_{T_c}^+, \mathcal{A}(\mathbb{C}^{2\nu}))$  allows one to consider the function  $p^{\text{conj}}$  as a solution of the Schrödinger equation, viewed as a limit case of the heat equation. Both spaces are local with respect to the space variables  $x$  and  $y$ . The space  $\mathcal{C}_{b,1}^\infty(i) - T_c, T_c[\times \mathbb{R}^{2\nu})$ , which gives information about global properties of the function  $p^{\text{conj}}$  with respect to the space variables, provides a unicity statement (see [Ha7]).

### 3 Proofs of the results

#### 3.1 Some properties of classical objects associated to $P_0$

Let us recall why equation (1.2) admits a solution as in (1.3). Let

$$S(x, y, t) := \int_0^t L|_{q=q_t^\natural} ds = \frac{1}{t} \int_0^1 \tilde{L}|_{q=\tilde{q}_t^\natural} ds = \frac{1}{t} \Phi(x, y, t) \quad (3.1)$$

where

$$\tilde{L} := \frac{1}{4} A^{-1}(ts) \cdot \dot{q} \otimes \dot{q} + t\dot{q} \cdot B(ts)q + t^2 C(ts) \cdot q \otimes q$$

and

$$\Phi(x, y, t) := \int_0^1 \tilde{L}|_{q=\tilde{q}_t^\natural} ds. \quad (3.2)$$

$\Phi$ , as  $\tilde{q}_t^\natural$ , is analytic for complex  $t$  with small modulus. Since  $\tilde{q}_t^\natural$  is linear with respect to  $x, y$ ,  $\Phi$  is a polynomial of total degree 2 in  $x, y$  and its coefficients are analytic near 0. By classical theory, since  $q_t^\natural$  verifies the Euler-Lagrange equations,  $S$  satisfies the eikonal equation

$$\partial_t S + H|_{q=x, p=\partial_x S} = 0,$$

where

$$H = \dot{q} \cdot \frac{\partial L}{\partial \dot{q}} - L = A \cdot (p - Bq) \otimes (p - Bq) - C \cdot q \otimes q.$$

Let us remark that

$$\partial_x S = p_t^\natural(t), \quad (3.3)$$

where  $p_t^\sharp := \frac{\partial L}{\partial \tilde{q}}|_{q=q_t^\sharp}$ .

Putting  $u = \lambda_t e^{-S}$  in (1.2) shows us that the partial differential equation in (1.2) is equivalent to

$$\partial_t \lambda_t = \left(-\frac{1}{t} A(t) \cdot \partial_x^2 \Phi + \gamma(t)\right) \times \lambda_t, \quad (3.4)$$

where  $\gamma(t) := \text{Tr}(A(t)B(t))$ . Differential equations and boundary conditions satisfied by  $\tilde{q}_t^\flat$  and  $\tilde{q}_t^\sharp$  involve that  $\tilde{q}_t^\sharp = y + s(x - y) + t\chi(s, t, x, y)$  where  $\chi$  is linear in  $x, y$  with analytic coefficients in  $s, t$ . Then

$$\Phi(x, y, t) = \frac{1}{4} A^{-1}(0) \cdot (x - y)^2 + t\Psi(t, x, y), \quad (3.5)$$

where  $\Psi$  is a polynomial of total degree bounded by 2, with respect to  $x, y$ , with analytic coefficients in  $t$ . Since  $\partial_x^2 \Phi$  only depends on  $t$ ,

$$-\frac{1}{t} A(t) \cdot \partial_x^2 \Phi + \gamma(t) = -\frac{\nu}{2t} + \theta(t)$$

where the function  $\theta$  is analytic near 0. Then for every  $\Delta > 0$  and every polynomial  $K \in \mathbb{C}[y]$ ,

$$\lambda_t = \frac{1}{(4\pi\Delta t)^{\nu/2}} \exp\left(\int_0^t \theta(s) ds + K(y)\right)$$

satisfies (3.4). Then, by (3.5), the function

$$u := \frac{1}{(4\pi\Delta t)^{\nu/2}} e^{\int_0^t \theta(s) ds - \Psi(t, x, y) + K(y)} e^{-\frac{1}{4t} A^{-1}(0) \cdot (x - y)^2}$$

is solution of the partial differential equation in (1.2). Let us choose

$$\Delta := \det(A_{j,k}(0))_{1 \leq j, k \leq \nu}.$$

Then

$$u|_{t=0+} = e^{-\Psi(0, y, y) + K(y)} \delta_{x=y}.$$

Let us choose  $K(y) = \Psi(0, y, y) = \frac{1}{t} \Phi(y, y, t)$ . Then  $u$  is the solution of (1.2) and, denoting this solution by  $p^0$ , (1.3) is satisfied where

$$\Phi_0(x, y, t) := \Phi(x, y, t) - \Phi(y, y, t) + t^2 \int_0^1 \theta(ts) ds. \quad (3.6)$$

The following results will be useful.

**Lemma 3.1** *There exists  $T_a \in ]0, \bar{T}[$  such that for every  $s \in [0, 1]$ ,  $(x, y) \in \mathbb{C}^{2\nu}$  and  $t \in D_{T_a}$ ,*

$$|\tilde{q}_t^\sharp(s)| \leq 2(|x| + |y|). \quad (3.7)$$



**Proof** By (2.8),  $|\tilde{q}_t^\flat(s)| \leq 2$  for complex  $t$  with small modulus. Similarly, the same estimate holds for  $\tilde{q}_t^\sharp$ . Then (2.5) implies (3.7).  $\square$

**Proposition 3.2** *For small positive number  $t$ , for every  $(x, y) \in \mathbb{C}^{2\nu}$ ,  $s \in [0, t]$  and  $\alpha \in \{1, \dots, \nu\}$*

$$\frac{1}{p^0}(\partial_x + B(t)x)p^0 = -\frac{1}{2}A^{-1}(t)\dot{q}_t^\flat(t). \quad (3.8)$$

$$(\partial_t + \dot{q}_t^\flat(t) \cdot \partial_x)q_{t,\alpha}^\flat(s) = 0, \quad (3.9)$$

$q_{t,\alpha}^\flat$  denoting the  $\alpha$ -coordinate of the vector  $q_t^\flat$ .

**Proof** Recall that  $p^0 := \frac{1}{(4\pi\Delta t)^{\nu/2}}e^{-\frac{1}{t}\Phi_0}$ . By (3.6),  $p^0 = \frac{1}{(4\pi\Delta t)^{\nu/2}}e^{-S+\Gamma(t,y)}$  where  $\Gamma$  is a polynomial in  $y$  with analytic coefficients in  $t$  near 0. Then, by (3.3),

$$\frac{1}{p^0}(\partial_x + B(t)x)p^0 = -\partial_x S + B(t)x = -p_t^\flat(t) + B(t)q_t^\flat(t).$$

But  $p = \frac{\partial L}{\partial \dot{q}} = \frac{1}{2}A^{-1}\dot{q} + Bq$ . This proves (3.8).

Let  $w(s) := (\partial_t + \dot{q}_t^\flat(t) \cdot \partial_x)q_t^\flat(s)$ . Hence  $w(0) = 0$  and  $w(t) = \left[\frac{\partial}{\partial t}q_t^\flat(s)\right]_{s=t} + \dot{q}_t^\flat(t)$  since  $q_t^\flat(t) = x$ . Then  $w(t) = \frac{d}{dt}(q_t^\flat(t)) = 0$ . Moreover  $q_t^\flat$ , and therefore  $w$ , is solution of (2.3) since the operator  $\partial_t + \dot{q}_t^\flat(t) \cdot \partial_x$  does not depend on  $s$ . For small  $t$ , the null function is the unique solution of (2.3) with vanishing boundary conditions. Then  $w \equiv 0$  and (3.9) is proven.  $\square$

**Remark 3.3** *the identity (3.9) is a generalization of [Ha4, (4.23)] and [Ha5, (3.18)].*

**Lemma 3.4** *Let  $A$ ,  $B$  and  $C$  as in Section 2 such that  $A$ ,  $iB$  and  $C$  satisfy (2.13). Then there exists  $T_e \in ]0, \bar{T}[$  such that*

1. *The matrix-valued functions  $\tilde{q}_t^\flat$  and  $\tilde{q}_t^\sharp$  are real for  $t \in i\mathbb{R}, |t| < T_e$ .*
2. *For  $s, s' \in [0, 1]$ , the coefficients of the matrix  $\tilde{K}_t(s, s')$  are real for  $t \in i\mathbb{R}, |t| < T_e$ .*
3. *There exist  $\Phi_1$  a polynomial with respect to  $x$  and  $y$  whose coefficients are analytic near 0 and  $k$  an analytical function near 0 such that*

$$p_t^0(x, y) := \frac{k(t)}{(4\pi\Delta t)^{\nu/2}}e^{-\frac{1}{t}\Phi_1(x, y, t)}$$

*and  $\Phi_1|_{x, y \in \mathbb{R}^\nu, t \in i] - T_e, T_e[}$  is  $\mathbb{R}$ -valued.*

**Proof** We take the point of view of the Schrödinger equation. Since the functions  $A, iB, C$  satisfy (2.13), the Lagrangian  $\tilde{L}|_{t=i\tilde{t}}$  is a polynomial with respect to  $\dot{q}$  and  $q$  with coefficients which are real functions with respect to  $\tilde{t}$ . Therefore the Euler equations associated to  $\tilde{L}|_{t=i\tilde{t}}$  by differentiating with respect to  $\tilde{t}$  have real coefficients and  $\tilde{q}_{i\tilde{t}}^\flat, \tilde{q}_{i\tilde{t}}^\sharp$  and  $\tilde{q}_{i\tilde{t}}^\natural$  (for  $x, y \in \mathbb{R}^\nu$ ) are real for  $\tilde{t} \in \mathbb{R}$ ,  $|\tilde{t}|$  small enough. Then, by (2.11),  $K_t(s, s') \in \mathbb{R}$  for  $s, s' \in [0, 1]$  and by (3.2),  $\Phi(x, y, t) \in \mathbb{R}$  for  $t \in i\mathbb{R}$ ,  $|t|$  small enough, and  $x, y \in \mathbb{R}^\nu$ . Let us choose  $\Phi_1(t, x, y) := \Phi(t, x, y) - \Phi(t, y, y)$ . Then by (3.6), Assertion 3 holds.  $\square$

### 3.2 The propagator equation

In a heuristic way, the shape of the deformation formula can be explained by the Wiener representation of the heat kernel and Wick's theorem (see [Ha4, Appendix]). The matrix  $K_t(s, s')$  appears therefore as a propagator. In this section, we prove that  $K_t(s, s')$  indeed satisfies the propagator equation (cf. Proposition 3.6). First, we claim that  $(q_t^\flat, p_t^\flat)$  satisfies Hamiltonian equations associated to a Hamiltonian

$$\mathcal{H} = \text{Tr}({}^t p L p + {}^t p M q + {}^t q N q) \quad (3.10)$$

where  $L, M, N$  are matrix-valued analytic functions near 0,  $L, N$  being symmetric. Let us introduce some notation. Let  $\mathcal{M}$  be a finite dimensional square-matrix space. If  $f$  is a regular  $\mathbb{R}$ -valued function on  $\mathcal{M}^2$ , we denote by  $\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}$  the matrices defined by

$$df(X, Y) \cdot (H, K) = \text{Tr}\left({}^t H \frac{\partial f}{\partial X}\right) + \text{Tr}\left({}^t K \frac{\partial f}{\partial Y}\right).$$

For instance  $\frac{\partial f}{\partial X} = BY$  and  $\frac{\partial f}{\partial Y} = {}^t BX$  if  $f(X, Y) = \text{Tr}({}^t XBY)$ . This notation will allow us to take into account the matrix structure of the trajectories of Lagrangian or Hamiltonian systems. In particular, we shall use the matrix product (see Lemma 3.5). The Euler-Lagrange equations associated to Lagrangian

$$\mathcal{L} := \text{Tr}\left(\frac{1}{4} {}^t \dot{q} A^{-1} \dot{q} + {}^t \dot{q} B q + {}^t q C q\right)$$

yields (2.6), which proves that  $q_t^\flat$  is a solution of these equations. Therefore  $p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{1}{2} A^{-1} \dot{q} + B q$  and the Lagrangian  $\mathcal{L}$  yields a Hamiltonian

$$\mathcal{H} = \text{Tr}\left({}^t \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \mathcal{L} = \text{Tr}\left(\frac{1}{4} {}^t \dot{q} A^{-1} \dot{q} - {}^t q C q\right) = \text{Tr}({}^t (p - B q) A (p - B q) - {}^t q C q).$$

Then  $(q_t^\flat, p_t^\flat)$  satisfies Hamiltonian equations with a Hamiltonian as in (3.10).

**Lemma 3.5** *Let  $\mathcal{H}$  be a Hamiltonian as in (3.10). Then the matrix  ${}^t q p - {}^t p q$  is constant along Hamiltonian trajectories.*

**Proof** It suffices to prove that  ${}^t\dot{q}p + {}^tq\dot{p}$  is symmetric on Hamiltonian trajectories. Since  $\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = 2Lp + Mq$  and  $\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = -(2Nq + {}^tMp)$ ,

$${}^t\dot{q}p + {}^tq\dot{p} = 2{}^tpLp - 2{}^tqNq.$$

This matrix is symmetric which proves Lemma 3.5.  $\square$

Now, we can prove that  $K_t(s, s')$  satisfies the propagator equation.

**Proposition 3.6** *For small positive number  $t$  and every  $(s, s') \in ]0, t]^2$ ,  $K_t(s, s')$  satisfies*

$$\begin{cases} -\frac{d^2 K}{ds^2} + E(s)\frac{dK}{ds} + F(s)K = A(s')\delta_{s=s'} \\ K|_{s=0} = K|_{s=t} = 0 \end{cases}.$$

**Proof** By (2.10) and since  $q_s^b(s) = \mathbb{1}$ ,

$$\frac{dK}{ds} = \int_{s \vee s'}^t \dot{q}_\tau^b(s) A(\tau) {}^tq_\tau^b(s') d\tau - 1_{s' < s} A(s) {}^tq_s^b(s')$$

and

$$\begin{aligned} \frac{d^2 K}{ds^2} &= \int_{s \vee s'}^t \ddot{q}_\tau^b(s) A(\tau) {}^tq_\tau^b(s') d\tau \\ &- 1_{s' < s} \left( \dot{q}_s^b(s) A(s) {}^tq_s^b(s') + \frac{d}{ds} (A(s) {}^tq_s^b(s')) \right) - A(s') \delta_{s=s'}. \end{aligned}$$

Then, since  $q_\tau^b$  satisfies (2.6),

$$-\frac{d^2 K}{ds^2} + E(s)\frac{dK}{ds} + F(s)K = A(s')\delta_{s=s'} + 1_{s' < s} w(s')$$

where

$$w(s') := \dot{q}_s^b(s) A(s) {}^tq_s^b(s') + \frac{d}{ds} (A(s) {}^tq_s^b(s')) - E(s) A(s) {}^tq_s^b(s').$$

We claim that  $w \equiv 0$ . Since  ${}^tw$  satisfies (2.6), it suffices to check that  $w|_{s'=0} = 0$  and  $w|_{s'=s} = 0$ . The first equality is obvious. Since  $q_s^b(s) = \mathbb{1}$ ,

$$\left[ \frac{\partial}{\partial s} q_s^b(s') \right] \Big|_{s'=s} = -\dot{q}_s^b(s).$$

Then

$$w|_{s'=s} = \dot{q}_s^b(s) A(s) + \dot{A}(s) - A(s) {}^t\dot{q}_s^b(s) - E(s) A(s).$$

By (2.4)

$$w|_{s'=s} = A(s) \left( A^{-1}(s) \dot{q}_s^b(s) - {}^t\dot{q}_s^b(s) A^{-1}(s) + 2(B(s) - {}^tB(s)) \right) A(s).$$

But  $p = \frac{\partial L}{\partial \dot{q}} = \frac{1}{2} A^{-1} \dot{q} + Bq$ . Then

$$w|_{s'=s} = 2A(s) (p_s^b(s) - {}^tp_s^b(s)) A(s).$$

By Lemma 3.5, the matrix  ${}^tq_s^b p_s^b - {}^tp_s^b q_s^b$  is constant. It vanishes for  $s' = 0$  and is equal to  $p_s^b(s) - {}^tp_s^b(s)$  for  $s' = s$ . Hence  $w|_{s'=s} = 0$  and  $w$  vanishes. This proves Proposition 3.6.  $\square$

### 3.3 The deformation matrix

For the proof of Theorem 2.1, we must establish some properties of  $\tilde{K}_t(s)$  (here  $s \in [0, 1]^n$ , cf. (2.16)) which we call the deformation matrix. We already studied it [Ha4] in a particular case. The following lemma (see [Ha4]) will be useful.

**Lemma 3.7** *Let  $\tilde{T} > 0$  and  $M > 0$ . There exists  $T > 0$  satisfying the following property. Let  $f$  be an analytic function on  $D_{\tilde{T}}$  verifying  $f(0) = 0$ ,  $f'(0) = 1$ ,  $\sup_{t \in D_{\tilde{T}}} |f(t)| \leq M$  and, for every  $t \in D_{\tilde{T}}$ ,*

$$\operatorname{Re} t = 0 \Rightarrow \operatorname{Re} f(t) = 0. \quad (3.11)$$

*Then, for  $t \in D_T$ ,*

$$\operatorname{Re} t > 0 \Rightarrow \operatorname{Re} f(t) > 0. \quad (3.12)$$

**Proposition 3.8** *Let  $\mathcal{E}$  be the space of measures  $\mu = \sum_{j=1}^n \delta_{s_j} \xi_j$  such that  $n \geq 1$ ,  $\xi_j \in \mathbb{R}^\nu$ ,  $s_j \in ]0, 1[$ . For complex  $t$  with small modulus, we denote by  $(\cdot, \cdot)_t$  the following bilinear form on  $\mathcal{E}$*

$$(\mu_1, \mu_2)_t := \int_0^1 \int_0^1 d\mu_1(s) \cdot \tilde{K}_t(s, s') d\mu_2(s').$$

*Notice that*

$$(\mu_1, \mu_2)_0 = \int_0^1 \int_0^1 s \wedge s' (1 - s \vee s') A(0) \cdot d\mu_1(s) \otimes d\mu_2(s').$$

*Then for complex  $t$  with small modulus*

$$\forall \mu \in \mathcal{E}, |(\mu, \mu)_t| \leq 2(\mu, \mu)_0. \quad (3.13)$$

**Remark 3.9** *The bilinear form  $(\cdot, \cdot)_0$  is symmetric positive definite (see [Ha4, Rem. 4.4]).*

**Proof** By Proposition 3.6 and analytic continuation,  $\tilde{K}_t$  satisfies for complex  $t$  with small modulus

$$\begin{cases} A^{-1}(ts) \left( -\frac{d^2}{ds^2} + tE(ts) \frac{d}{ds} + t^2 F(ts) \right) \tilde{K}_t(s, s') = \delta_{s=s'} \\ \tilde{K}_t|_{s=0} = \tilde{K}_t|_{s=1} = 0 \end{cases}. \quad (3.14)$$

Let  $(\xi_1, \dots, \xi_n) \in \mathbb{R}^{\nu n}$  and  $(s_1, \dots, s_n) \in ]0, 1[^n$ . The function  $u$  defined by

$$u(s) = \sum_{j=1}^n \tilde{K}_t(s, s_j) \xi_j$$

is continuous and piecewise differentiable on  $[0, 1]$ . Let  $\mu := \sum_{j=1}^n \delta_{s_j} \xi_j$ . By (3.14)

$$\begin{cases} A^{-1}(ts) \left( -\frac{d^2}{ds^2} + tE(ts) \frac{d}{ds} + t^2 F(ts) \right) u = \mu \\ u(0) = u(1) = 0 \end{cases} \quad (3.15)$$

Let  $H^0 := L^2([0, 1], \mathbb{C}^\nu)$ . Let  $\varepsilon_{k,l}$  be the coordinates of an orthonormal basis of  $\mathbb{R}^\nu$  diagonalizing the real symmetric matrix  $A^{-1}(0)$ . For  $n \geq 1$ ,  $k, l \in \{1, \dots, \nu\}$  and  $s \in [0, 1]$ , set  $e_{n,k,l}(s) = \sqrt{2} \sin(n\pi s) \varepsilon_{k,l}$ .  $(e_{n,k})_{n,k}$  is an orthonormal basis of  $H^0$  which diagonalizes the unbounded self-adjoint operator  $S := -A^{-1}(0) \frac{d^2}{ds^2}$  (Dirichlet boundary conditions). Let

$$\begin{aligned} H_0^1 &:= \left\{ f \in H^0 \left| \frac{df}{ds} \in L^2, f(0) = f(1) = 0 \right. \right\} \\ &= \left\{ \sum_{n,k} f_{n,k} e_{n,k} \left| \sum_{n,k} |n f_{n,k}|^2 < \infty \right. \right\} \end{aligned}$$

and

$$H^{-1} := \left\{ \sum_{n,k} f_{n,k} e_{n,k} \left| \sum_{n,k} \left| \frac{f_{n,k}}{n} \right|^2 < \infty \right. \right\}.$$

For  $(f, g) \in H^{-1} \times H_0^1$  or  $(f, g) \in H^0 \times H^0$ , let

$$\langle f, g \rangle := \int_0^1 \bar{f}(s) \cdot g(s) ds = \sum_{n,k} \bar{f}_{n,k} g_{n,k}.$$

The operator  $S^{1/2}$  can be viewed as an isomorphism from  $H_0^1$  to  $H^0$  and from  $H^0$  to  $H^{-1}$ . Natural Hilbertian norms induced by  $S^{\pm 1/2}$  can be defined on  $H_0^1$  and  $H^{-1}$ . Then  $\mu \in H^{-1}$  and

$$\mu = (S + T_t)u,$$

where

$$T_t = -(A^{-1}(ts) - A^{-1}(0)) \frac{d^2}{ds^2} + A^{-1}(ts) \left( tE(ts) \frac{d}{ds} + t^2 F(ts) \right).$$

Since

$$S + T_t = S^{1/2} (1 + S^{-1/2} T_t S^{-1/2}) S^{1/2}$$

and  $\|S^{-1/2} T_t S^{-1/2}\|_{L(H^0, H^0)}$  goes to 0 when  $t$  goes to 0, one has, for small complex  $t$ , that  $1 + S^{-1/2} T_t S^{-1/2}$  is invertible and

$$\|(1 + S^{-1/2} T_t S^{-1/2})^{-1}\|_{L(H^0, H^0)} \leq 2.$$

Hence

$$\begin{aligned} (\mu, \mu)_t &= \langle \mu, u \rangle \\ &= \langle \mu, S^{-1/2} (1 + S^{-1/2} T_t S^{-1/2})^{-1} S^{-1/2} \mu \rangle \\ &= \langle S^{-1/2} \mu, (1 + S^{-1/2} T_t S^{-1/2})^{-1} S^{-1/2} \mu \rangle \end{aligned}$$

and by Cauchy-Schwarz inequality

$$|(\mu, \mu)_t| \leq 2 \|S^{-1/2} \mu\|_{H^0}^2.$$

But  $|S^{-1/2} \mu|_{H^0}^2 = \langle \mu, S^{-1} \mu \rangle = (\mu, \mu)_0$ . This proves (3.13).  $\square$

**Proposition 3.10** *Let  $A$ ,  $B$  and  $C$  be as in Theorem 2.1. There exists  $T_d \in ]0, \tilde{T}[$  such that for every  $n \geq 1$ ,  $s = (s_1, \dots, s_n) \in [0, 1]^n$ ,  $(\xi_1, \dots, \xi_n) \in \mathbb{R}^{\nu^n}$ ,  $t \in \mathbb{C}$  with the condition  $0 < s_1 < \dots < s_n < 1$ ,  $|t| < T_d$ ,*

$$\operatorname{Re} t \geq 0 \Rightarrow \operatorname{Re}(t \tilde{\mathbf{K}}_t(s) \cdot \xi \otimes_n \xi) \geq 0, \quad (3.16)$$

$$|\tilde{\mathbf{K}}_t(s) \cdot \xi \otimes_n \xi| \leq 2n|A(0)|_\infty \sum_{j=1}^n \xi_j^2. \quad (3.17)$$

**Proof** Let  $\mu := \sum_{j=1}^n \delta_{s_j} \xi_j$ . Then  $\tilde{\mathbf{K}}_t(s) \cdot \xi \otimes_n \xi = (\mu, \mu)_t$  and

$$(\mu, \mu)_0 = \sum_{j,k=1}^n s_{j \wedge k} (1 - s_{j \vee k}) \xi_j \cdot A(0) \xi_k.$$

Then  $(\mu, \mu)_0 \leq n|A(0)|_\infty \sum_{j=1}^n \xi_j^2$ . Hence Proposition 3.8 implies (3.17) if  $T_d$  is small enough.

Let us choose arbitrary vectors  $\xi_1, \dots, \xi_n$  such that  $(\xi_1, \dots, \xi_n)$  does not vanish. By Remark 3.9,  $(\mu, \mu)_0 \neq 0$ . Let  $f$  be the function defined by

$$f(t) = t \frac{(\mu, \mu)_t}{(\mu, \mu)_0} = t \frac{\tilde{\mathbf{K}}_t(s) \cdot \xi \otimes_n \xi}{(\mu, \mu)_0}.$$

We claim that the function  $f$  satisfies (3.11). It suffices to check that  $g(t) := \tilde{\mathbf{K}}_t(s) \cdot \xi \otimes_n \xi$  satisfies (2.13). This holds by Lemma 3.4. By (3.13),  $f$  is bounded for complex  $t$  with small modulus. Obviously  $f(0) = 1$  and  $f'(0) = 1$ . By Lemma 3.7, there exists  $T_d > 0$  such that  $\operatorname{Re} f(t) > 0$  for  $t \in D_{T_d}^+$ . Since  $(\mu, \mu)_0 \in ]0, +\infty[$ , (3.16) holds for  $t \in D_{T_d}$ .  $\square$

**Remark 3.11** *The reality assumption (2.13) is crucial for establishing (3.11). What happens when (2.13) does not hold? Then the statement of Lemma 3.7 can be replaced by the following one. Let  $\tilde{T} > 0$ ,  $M > 0$  and  $\varepsilon \in ]0, \pi/2[$ . There exists  $T_\varepsilon > 0$  such that every analytic function  $f$  on  $D_{\tilde{T}}$ , with  $f(0) = 0$ ,  $f'(0) = 1$ ,  $\sup_{t \in D_{\tilde{T}}} |f(t)| \leq M$ , satisfies*

$$\begin{cases} t \in D_{T_\varepsilon} \\ \arg t \in ]-\pi/2 + \varepsilon, \pi/2 - \varepsilon[ \end{cases} \Rightarrow \operatorname{Re} f(t) > 0.$$

Therefore (3.16) can be replaced by

$$\begin{cases} t \in D_{T_\varepsilon} \\ \arg t \in ]-\pi/2 + \varepsilon, \pi/2 - \varepsilon[ \end{cases} \Rightarrow \operatorname{Re}(t \tilde{\mathbf{K}}_t(s) \cdot \xi \otimes_n \xi) \geq 0.$$

Then, even if Assumption (2.13) is removed, the deformation formula will remain valid for  $t \in D_{T_\varepsilon} - \{0\}$ ,  $\arg t \in ]-\pi/2 + \varepsilon, \pi/2 - \varepsilon[$ . One can expect to recover the Schrödinger kernel if the function  $c$  is chosen as in [Ha7, Proposition 4.5 (case 2)].

### 3.4 Proof of Theorem 2.1

The following lemma will be useful.

**Lemma 3.12** *Let  $m \geq 0$  and  $\Omega_1, \Omega_2$  be some open subsets of  $\mathbb{C}$  such that  $\Omega_1 \subset \subset \Omega_2$ . There exists  $C_{m,\Omega_1,\Omega_2} > 0$  satisfying the following property: for every analytic bounded matrix-valued function  $\theta$  on  $\Omega_2$  and every analytic bounded  $\mathbb{C}$ -valued function  $\varphi$  on  $\Omega_2$  one has*

$$\partial_t^m(\theta e^\varphi) = \alpha_m e^\varphi \quad (3.18)$$

where  $\alpha_m$  denotes an analytic matrix-valued function on  $\Omega_2$  such that

$$\sup_{\Omega_1} |\alpha_m| \leq C_{m,\Omega_1,\Omega_2} \times \sup_{\Omega_2} |\theta| \times (1 + (\sup_{\Omega_2} |\varphi|)^m).$$

**Proof** The lemma can be proved with the help of Cauchy's formula by induction on  $m$ .  $\square$

Let us prove Theorem 2.1. We choose  $T_c = \frac{1}{2} \min(1, T_a, T_b, T_d, T_e)$  (see Lemma 3.1, (2.12), Proposition 3.10 and Lemma 3.4).

**-1-** Let us check that  $v_n$  given by (2.17) and  $p^{\text{conj}} = \mathbb{1} + \sum_{n \geq 1} v_n$  are well defined for  $t \in D_{T_c}^+$ . For  $t \in D_{2T_c}^+$ , let

$$\varphi_n(t) := \tilde{q}_t^\sharp(s) \cdot \xi + it \tilde{K}_t(s) \cdot \xi \otimes_n \xi,$$

$$F_n := t^n e^{i\varphi_n(t)} f(s_n t, \xi_n) \cdots f(s_1 t, \xi_1).$$

Let  $R > 0$  and let  $(x, y) \in \mathbb{C}^{2\nu}$  such that  $|x| + |y| < R$ . By Lemma 3.1 and by (3.16),  $|i\tilde{q}_t^\sharp(s) \cdot \xi| \leq 2R(|\xi_1| + \cdots + |\xi_n|)$  and  $\text{Re}(t\tilde{K}_t(s) \cdot \xi \otimes_n \xi) \geq 0$ . For  $\xi^* \in \mathbb{R}^\nu$  and  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^{\nu n}$ , let us denote

$$\mathfrak{f}(\xi^*) := \sup_{|t| < T_b} |f(t, \xi^*)|, \mathfrak{f}^\otimes(\xi) := \mathfrak{f}(\xi_n) \cdots \mathfrak{f}(\xi_1),$$

$$d^{\nu n} \mu_*^\otimes(\xi) := d\mu_*(\xi_n) \cdots d\mu_*(\xi_1),$$

$$G_n := 2^n T_c^n \exp(2R(|\xi_1| + \cdots + |\xi_n|)) \mathfrak{f}^\otimes(\xi).$$

Then  $|F_n| \leq G_n$ . Let

$$A := 2 \int_{\mathbb{R}^\nu} \exp(2R|\xi|) \mathfrak{f}(\xi) d\mu_*(\xi).$$

Then

$$\int_{\mathbb{R}^{\nu n}} G_n d^{\nu n} \mu_*^\otimes(\xi) \leq (AT_c)^n$$

---

<sup>1</sup>i.e. there exists  $\rho > 0$  such that  $\Omega_1 + D_\rho \subset \Omega_2$ .

and

$$\sum_{n \geq 1} \int_{0 < s_1 < \dots < s_n < 1} \int_{\mathbb{R}^{\nu n}} G_n d^{\nu n} \mu_*^\otimes(\xi) d^n s < \infty.$$

Hence  $v_n$  and  $p^{\text{conj}}$  are well defined on  $D_{2T_c}^+ \times \mathbb{C}^{2\nu}$  since  $R$  is arbitrary (let us remark that the expressions (2.15) and (2.17) of  $v_n$  are clearly equivalent). By the dominated convergence theorem,  $p^{\text{conj}} \in \mathcal{A}(D_{T_c}^+ \times \mathbb{C}^{2\nu})$ .

**-2-** Let us check that  $p^{\text{conj}}$  is well defined for  $t \in \bar{D}_{T_c}^+$  and that  $p^{\text{conj}} \in \mathcal{C}^\infty(\bar{D}_{T_c}^+, \mathcal{A}(\mathbb{C}^{2\nu}))$ . Let  $R > 0$ . By (3.7) and (3.17), there exists  $M > 0$  such that for  $n \geq 1$ ,  $0 < s_1 < \dots < s_n < 1$ ,  $(\xi_1, \dots, \xi_n) \in \mathbb{R}^{\nu n}$ ,  $t \in D_{2T_c}$ ,  $(x, y) \in \mathbb{C}^{2\nu}$  and  $|x| + |y| < R$ ,

$$|\varphi_n(t)| \leq M(1 + n(|\xi_1| + \dots + |\xi_n|)^2). \quad (3.19)$$

We want to use the dominated convergence theorem. For  $m \geq 0$ , let

$$F_{n,m} := \partial_t^m (t^n f(s_n t, \xi_n) \dots f(s_1 t, \xi_1) e^{i\varphi_n(t)}).$$

By Lemma 3.12 and (3.19), there exists  $K_m > 0$  such that

$$|F_{n,m}| \leq K_m^{n+1} n^m (1 + (|\xi_1| + \dots + |\xi_n|)^{2m}) \exp(2R(|\xi_1| + \dots + |\xi_n|)) \mathfrak{f}^\otimes(\xi),$$

for  $t \in D_{T_c}$ ,  $Rt \geq 0$ . Then the inequality

$$\frac{1}{(2m)!} (1 + (|\xi_1| + \dots + |\xi_n|)^{2m}) \leq \exp(|\xi_1| + \dots + |\xi_n|)$$

yields  $|F_{n,m}| \leq G_{n,m}$  where

$$G_{n,m} := (2m)! K_m^{n+1} n^m \exp((1 + 2R)(|\xi_1| + \dots + |\xi_n|)) \mathfrak{f}^\otimes(\xi).$$

Let

$$A = \int_{\mathbb{R}^\nu} \exp((1 + 2R)|\xi|) \mathfrak{f}(\xi) d\mu_*(\xi).$$

Then

$$\int_{\mathbb{R}^{\nu n}} G_{n,m} d^{\nu n} \mu_*^\otimes(\xi) \leq (2m)! n^m K_m^{n+1} A^n$$

and

$$\sum_{n \geq 1} \int_{0 < s_1 < \dots < s_n < 1} \int_{\mathbb{R}^{\nu n}} G_{n,m} d^{\nu n} \mu_*^\otimes(\xi) d^n s < \infty.$$

Since  $R$  and  $m$  are arbitrary, the dominated convergence theorem proves that  $p^{\text{conj}} \in \mathcal{C}^\infty(\bar{D}_{T_c}^+, \mathcal{A}(\mathbb{C}^{2\nu}))$ .

**-3-** Let us check that  $p^{\text{conj}} \in \mathcal{C}_{b,1}^\infty(i] - T_c, T_c[ \times \mathbb{R}^{2\nu})$ . Let  $\alpha \in \mathbb{N}$ ,  $\beta, \gamma \in \mathbb{N}^\nu$ ,  $x, y \in \mathbb{R}^\nu$  and  $\tilde{t} \in ]-2T_c, 2T_c[$ . Let

$$F_n := \partial_{\tilde{t}}^\alpha \partial_x^\beta \partial_y^\gamma ((i\tilde{t})^n e^{i\varphi_n(i\tilde{t})} f(i\tilde{t}s_n, \xi_n) \dots f(i\tilde{t}s_1, \xi_1)).$$



Let  $(e_1, \dots, e_\nu)$  be the standard basis of  $\mathbb{R}^\nu$ . For  $\delta = 1, \dots, \nu$ , let us denote

$$\varpi_{\delta, \flat}(t) := (\tilde{q}_t^\flat(s_1)e_\delta) \cdot \xi_1 + \dots + (\tilde{q}_t^\flat(s_n)e_\delta) \cdot \xi_n,$$

$$\varpi_{\delta, \sharp}(t) := (\tilde{q}_t^\sharp(s_1)e_\delta) \cdot \xi_1 + \dots + (\tilde{q}_t^\sharp(s_n)e_\delta) \cdot \xi_n.$$

Then

$$F_n = i^\alpha \partial_t^\alpha (\theta_n(t) e^{i\varphi_n(t)}) \Big|_{t=i\tilde{t}}$$

where

$$\theta_n(t) := t^n \varpi_{1, \flat}^{\beta_1}(t) \cdots \varpi_{\nu, \flat}^{\beta_\nu}(t) \varpi_{1, \sharp}^{\gamma_1}(t) \cdots \varpi_{\nu, \sharp}^{\gamma_\nu}(t) f(ts_1, \xi_1) \cdots f(ts_n, \xi_n).$$

Then there exists  $M_1 > 0$  such that, for  $t \in D_{2T_c}$ ,

$$|\theta_n(t)| \leq M_1 |\xi|_1^{|\beta|+|\gamma|} \mathfrak{f}^\otimes(\xi).$$

By (3.17) there exists  $M_2 > 0$ , such that, for  $t \in D_{2T_c}$ ,

$$|\varphi_n(t)| \leq M_2 (|x| + |y|) |\xi|_1 + n |\xi|_1^2.$$

Then, by Lemma 3.12, there exists  $C > 0$  such that for  $\tilde{t} \in ]-T_c, T_c[$ ,  $n \geq 1$ ,  $\xi \in \mathbb{R}^{\nu n}$  and  $x, y \in \mathbb{R}^\nu$

$$|F_n| \leq C \left( 1 + (|x| + |y|) |\xi|_1 + n |\xi|_1^2 \right)^\alpha |\xi|_1^{|\beta|+|\gamma|} \mathfrak{f}^\otimes(\xi).$$

Here we also use that, by assertions 1 and 2 of Lemma 3.4,  $\varphi_n(i\tilde{t}) \in \mathbb{R}$ . Then, by binomial formula, there exists  $C' > 0$  such that

$$\begin{aligned} |F_n| &\leq C' \times \left( 1 + \sum_{\alpha_1 + \alpha_2 = \alpha} (|x| + |y|)^{\alpha_1} |\xi|_1^{\alpha_1 + 2\alpha_2} n^{\alpha_2} \right) \times |\xi|_1^{|\beta|+|\gamma|} \mathfrak{f}^\otimes(\xi) \\ &\leq Q(x, y) n^\alpha e^{|\xi|_1} \mathfrak{f}^\otimes(\xi) \end{aligned}$$

where

$$Q(x, y) := C' \left( 1 + \sum_{\alpha_1 + \alpha_2 = \alpha} (\alpha_1 + 2\alpha_2 + |\beta| + |\gamma|)! (|x| + |y|)^{\alpha_1} \right).$$

Let

$$A = \int_{\mathbb{R}^\nu} e^{|\xi|} \mathfrak{f}(\xi) d\mu_*(\xi).$$

Then

$$\int_{\mathbb{R}^{\nu n}} |F_n| d^{\nu n} \mu_*^\otimes(\xi) d^n s \leq Q(x, y) n^\alpha A^n.$$

Therefore, for  $t \in i] - T_c, T_c[$  and  $x, y \in \mathbb{R}^\nu$ ,

$$|\partial_t^\alpha \partial_x^\beta \partial_y^\gamma u| \leq 1 + Q(x, y) \sum_{n \geq 1} n^\alpha \frac{A^n}{n!}.$$

This proves that  $p^{\text{conj}} \in \mathcal{C}_{b,1}^\infty(i) - T_c, T_c[\times \mathbb{R}^{2\nu})$ .

**-4-** Let us verify that the function  $p^0 p^{\text{conj}}$ , with  $p^{\text{conj}}$  given by (2.14), is a solution of (1.4). By continuity and analyticity arguments, it suffices to check (1.4) for small positive number  $t$ . Let

$$D_x := \partial_x + B(t)x.$$

Then, if  $v = v(t, x)$  is a regular function with respect to its arguments,

$$A(t) \cdot D_x^2(p^0 v) = (A(t) \cdot D_x^2 p^0)v + 2A(t) \cdot D_x p^0 \otimes \partial_x v + p^0 A(t) \cdot \partial_x^2 v.$$

A solution  $u$  of (1.4) is then given, if we use the relation  $u = p^0 v$ , by a solution  $v$  of the conjugate equation

$$\begin{cases} (\partial_t - \frac{2}{p^0} A(t) \cdot D_x p^0 \otimes \partial_x)v = A(t) \cdot \partial_x^2 v + c(t, x)v & (t \neq 0) \\ v|_{t=0^+} = \mathbb{1} \end{cases}.$$

Let  $v_0 = \mathbb{1}$ . By (3.8), it suffices to verify that, for  $n \geq 1$ ,  $v_n$  given by (2.15) satisfies

$$\begin{cases} (\partial_t + \dot{q}_t^\natural(t) \cdot \partial_x)v_n = A(t) \cdot \partial_x^2 v_n + c(t, x)v_{n-1} \\ v_n|_{t=0^+} = 0 \end{cases}, \quad (3.20)$$

for small positive number  $t$  and  $n \geq 1$ . By (2.15)

$$v_n = \int_{0 < s_1 < \dots < s_n < t} F_n d^n s,$$

where

$$F_n = \left[ \exp(\mathbf{K}_t(s) \cdot \partial_z \otimes_n \partial_z) c(s_n, z_n) \cdots c(s_1, z_1) \right] \Big|_{\substack{z_1 = q_t^\natural(s_1) \\ \vdots \\ z_n = q_t^\natural(s_n)}}.$$

Here

$$\mathbf{K}_t(s) \cdot \partial_z \otimes_n \partial_z := \sum_{j,k=1}^n \partial_{z_j} \cdot K_t(s_j, s_k) \partial_{z_k}$$

where  $K_t$  is defined by (2.10). Then

$$(\partial_t + \dot{q}_t^\natural(t) \cdot \partial_x)v_n = (\text{boundary}) + (\text{interior})$$

where

$$\begin{aligned} (\text{boundary}) &= \int_{0 < s_1 < \dots < s_{n-1} < t} F_n|_{s_n=t} d^{n-1} s, \\ (\text{interior}) &= \int_{0 < s_1 < \dots < s_n < t} (\partial_t + \dot{q}_t^\natural(t) \cdot \partial_x) F_n d^n s. \end{aligned}$$

Since

$$\mathbf{K}_t(s_1, \dots, s_{n-1}, t) \cdot \partial_z \otimes_n \partial_z = \mathbf{K}_t(s_1, \dots, s_{n-1}) \cdot \partial_z \otimes_{n-1} \partial_z,$$

one gets

$$(\text{boundary}) = c(t, x) v_{n-1}.$$

Now we claim that  $(\text{interior}) = A(t) \cdot \partial_x^2 v_n$ . By (3.9), if  $\varphi(z_1, \dots, z_n)$  is an arbitrary differentiable function of  $(z_1, \dots, z_n) \in \mathbb{C}^{\nu n}$ ,

$$(\partial_t + q_t^\sharp(t) \cdot \partial_x)[\varphi(z_1, \dots, z_n)] \Big|_{\substack{z_1 = q_t^\sharp(s_1) \\ \dots \\ z_n = q_t^\sharp(s_n)}} = 0.$$

Then

$$(\text{interior}) = \int_{0 < s_1 < \dots < s_n < t} G_n d^n s,$$

where

$$G_n = \left[ \partial_t (\mathbf{K}_t(s) \cdot \partial_z \otimes_n \partial_z) \exp(\mathbf{K}_t(s) \cdot \partial_z \otimes_n \partial_z) c(s_n, z_n) \cdots c(s_1, z_1) \right] \Big|_{\substack{z_1 = q_t^\sharp(s_1) \\ \dots \\ z_n = q_t^\sharp(s_n)}}.$$

On the other hand,

$$A(t) \cdot \partial_x^2 v_n = \int_{0 < s_1 < \dots < s_n < t} H_n d^n s$$

where

$$H_n = A(t) \cdot \partial_x^2 \left[ \exp(\mathbf{K}_t(s) \cdot \partial_z \otimes_n \partial_z) c(s_n, z_n) \cdots c(s_1, z_1) \right] \Big|_{\substack{z_1 = q_t^\sharp(s_1) \\ \dots \\ z_n = q_t^\sharp(s_n)}}.$$

For  $\bar{s} \in [0, t]$ ,  $q_t^\sharp(\bar{s}) = q_t^\flat(\bar{s})x + q_t^\sharp(\bar{s})y$ . Then

$$H_n = \left[ \sum_{j,k=1}^n A(t) \cdot ({}^t q_t^\flat(s_j) \partial_{z_j} \otimes {}^t q_t^\flat(s_k) \partial_{z_k}) \exp(\mathbf{K}_t(s) \cdot \partial_z \otimes_n \partial_z) \times \right. \\ \left. c(s_n, z_n) \cdots c(s_1, z_1) \right] \Big|_{\substack{z_1 = q_t^\sharp(s_1) \\ \dots \\ z_n = q_t^\sharp(s_n)}}.$$

By (2.10),  $\partial_t K_t(s, s') = q_t^\flat(s) A(t) {}^t q_t^\flat(s')$  and then

$$\partial_t (\mathbf{K}_t(s) \cdot \partial_z \otimes_n \partial_z) = \sum_{j,k=1}^n \partial_{z_j} \cdot q_t^\flat(s_j) A(t) {}^t q_t^\flat(s_k) \partial_{z_k}.$$

Therefore  $G_n = H_n$  and

$$(\text{interior}) = A(t) \cdot \partial_x^2 v_n.$$

Then (3.20) holds and  $p^0 p^{\text{conj}}$  satisfies (1.4).

## REFERENCES

- [A-H] S. A. Albeverio, R. J. Hoegh-Krohn, Mathematical Theory of Feynman path integrals, Lecture Notes in Mathematics **523** (1976).
- [B-B] R. Balian and C. Bloch, Solutions of the Schrödinger equation in terms of classical paths, Ann. of Phys. **85** (1974), 514-545.
- [Ge-Ya] I.M. Gel'fand, A.M. Yaglom, Integration in functional spaces and its applications in quantum physics, Journal of Mathematical Physics, 1-1 (1960), 48-69.
- [Ha4] T. Hargé, Borel summation of the small time expansion of the heat kernel. The scalar potential case (2013).
- [Ha7] Some remarks on the complex heat kernel on  $\mathbb{C}^\nu$  in the scalar potential case (2013).
- [Ha5] T. Hargé, Borel summation of the heat kernel with a vector potential case (2013).
- [It] K. Ito, Generalized uniform complex measures in the Hilbertian metric space with their applications to the Feynman integral, Fifth Berkeley Symp. on Math. Statist. and Prob. **2** (1967), 145–161.
- [On] E. Onofri, On the high-temperature expansion of the density matrix, American Journal Physics, 46-4 (1978), 379-382.
- [Ru] W. Rudin, Real and complex analysis, section 6.

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